Problem 1.37

Find formulas for r, θ, ϕ in terms of x, y, z (the inverse, in other words, of Eq. 1.62).

Solution

Eq. 1.62 gives the formulas to switch from Cartesian coordinates (x, y, z) into spherical coordinates (r, ϕ, θ), θ being the angle from the polar axis.

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \tag{1.62}$$

Square both sides of each equation and add the respective sides together.

$$\begin{aligned} x^2 + y^2 + z^2 &= (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2 \\ &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta \\ &= r^2\sin^2\theta + r^2\cos^2\theta \\ &= r^2(\sin^2\theta + \cos^2\theta) \\ &= r^2 \end{aligned}$$

Take the square root of both sides, choosing the positive root since the distance from the origin is positive.

$$r = \sqrt{x^2 + y^2 + z^2}$$

Eliminate r and θ by dividing the respective sides of the first two equations.

$$\frac{y}{x} = \frac{r\sin\theta\sin\phi}{r\sin\theta\cos\phi} = \tan\phi \quad \rightarrow \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

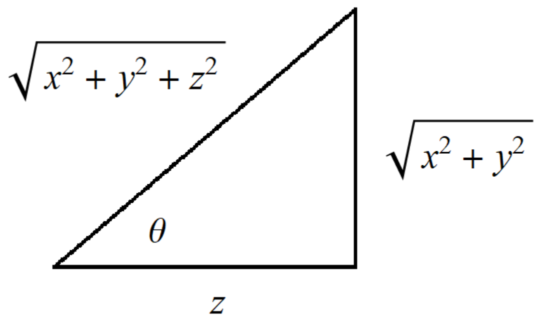
Solve the third equation for θ.

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \rightarrow \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Therefore, the formulas to switch from spherical coordinates (r, ϕ, θ) to Cartesian coordinates (x, y, z) are

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \\ \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{cases}.$$

The formula for θ can be written in terms of the inverse tangent as well. Draw the implied right triangle and determine the missing side using the Pythagorean theorem.



Consequently,

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} \quad \rightarrow \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$